SOLUTION OF SOME DUAL EQUATIONS ENCOUNTERED IN PROBLEMS OF THE THEORY OF ELASTICITY

PMM Vol. 31, No. 4, 1967, pp. 678-689

A.A. BABLOIAN
(Erevan)

(*Received November 19, 1966*)

The author presents some formulas for expansion of an arbitrary function into a series in terms of functions

$$\chi_k(t) = \begin{cases} e^{\alpha t} & (k=0) \\ \alpha \sin \gamma_k t + \gamma_k \cos \gamma_k t & (k=1,\ 2,\ \ldots) \end{cases} \tag{0.1}$$

$$\eta_k(t) = \begin{cases} e^{\alpha t} & (k=0) \\ \alpha \cos \gamma_k t - \gamma_k \sin \gamma_k t & (k=1,\ 2,\ \ldots) \end{cases} \tag{0.2}$$

$$y_k(x) = P_{k-1}(x) + P_k(x), \quad z_k(x) = P_{k-1}(x) - P_k(x)$$
$$\gamma_k = k\pi/t_1, \qquad 0 \leqslant t \leqslant t_1, \qquad -1 \leqslant x \leqslant 1 \tag{0.3}$$

where $P_k(x)$ are Legendre polynomials and α is a given number.

Functions $\eta_k(t)$ and $\chi_k(t)$ appear in the course of solving the plane problem of the theory of elasticity for an annular sector, in a problem of torsion of a conical shaft, etc., when solutions obtained are in the form of Fourier series and boundary conditions are satisfied exactly on the lines $\theta =$ const. In the case of a plane problem for an annular sector we have $\alpha = 1$, while in the case of torsion of a shaft we have $\alpha = 3/2$.

Investigation of functions $y_k(x)$ and $z_k(x)$ resulted from the necessity of obtaining solutions to dual equations containing functions $\chi_k(t)$ of the following form:

$$a_0\chi_0(t) + \sum_{k=1}^{\infty} \gamma_k^{\pm 1} a_k \chi_k(t) = f(t) \qquad (0 < t < \beta)$$

$$c a_0 \chi_0(t) + \sum_{k=1}^{\infty} a_k \chi_k(t) = g(t) \qquad (\beta < t < t_1) \tag{0.4}$$

where $f(t)$ and $g(t)$ are given functions, c is a known number and coefficients a_k remain to be determined.

1. In [1 and 2] it was shown that a set of functions $\{\chi_k(t)\}$ forms a closed orthogonal system in the interval $0 < t < t_1$ amongst functions satisfying Dirichlet conditions. This implies that a function $f(t) \in L_2(0, t_1)$ can be expanded into a Fourier series in terms of $\chi_k(t)$, and that we shall have

$$f(t) = a_0\chi_0(t) + \sum_{k=1}^{\infty} a_k \chi_k(t) \qquad (0 < t < t_1) \tag{1.1}$$

at the points of continuity of $f(t)$. Here coefficients of expansion are given by

$$a_0 = \frac{2\alpha}{e^{2\alpha t_1} - 1} \int_0^{t_1} f(t)\, \chi_0(t)\, dt, \quad a_k = \frac{2}{t_1(\gamma_k^2 + \alpha^2)} \int_0^{t_1} f(t)\, \chi_k(t)\, dt \tag{1.2}$$

To obtain (1.2) we have utilised the following value of the integral:

$$\int_0^{t_1} \chi_k(t)\,\chi_p(t)\,dt = \begin{cases} {}^1\!/_2 t_1(\gamma_k^2+\alpha^2) & (k=p\neq 0) \\ {}^1\!/_2\alpha^{-1}(e^{2\alpha t_1}-1) & (k=p=0) \\ 0 & (k\neq p) \end{cases} \tag{1.3}$$

Functions $\eta_k(t)$ are almost orthogonal since, when $k, p \neq 0$,

$$\int_0^{t_1} \eta_k(t)\,\eta_p(t)\,dt = \begin{cases} -\alpha\,[1-(-1)^{p+k}] & (p\neq k) \\ {}^1\!/_2 t_1(\gamma_k^2+\alpha^2) & (p=k) \end{cases} \tag{1.4}$$

Despite this, we have a following expansion

$$f(t) = b + b_0 e^{-\alpha(t_1-t)} + \sum_{k=1}^{\infty} b_k \eta_k(t) \tag{1.5}$$

Here $f(t)\in L_2(0, t_1)$ and unknown coefficients are uniquely given by

$$b = \frac{1}{t_1}\int_0^{t_1} f(x)\,dx, \qquad b_0 = -\frac{\alpha}{\operatorname{sh}\alpha t_1}\int_0^{t_1} f(x)\,e^{\alpha x}dx$$

$$b_k = \frac{2}{t_1(\gamma_k^2+\alpha^2)}\int_0^{t_1} f(x)\,\eta_k(x)\,dx \tag{1.6}$$

If we now insert a_k from (1.2) into (1.1), put $\gamma_k = k\pi/t_1 = x$, $\pi/t_1 = dx$ and pass formally to the limit with $t_1 \to \infty$, we obtain the following integral transformation formula:

$$f(t) = -2\alpha e^{\alpha t} q(\alpha)\int_0^{\infty} f(x)\,e^{\alpha x}dx + \frac{2}{\pi}\int_0^{\infty}\frac{\chi(x,t)}{x^2+\alpha^2}dx\int_0^{\infty} f(y)\,\chi(x,y)\,dy \tag{1.7}$$

where

$$\chi(x,t) = \alpha\sin xt + x\cos xt, \qquad q(\alpha) = \begin{cases} 1 & (\alpha<0) \\ 0 & (\alpha\geqslant 0) \end{cases} \tag{1.8}$$

Second integral transformation formula

$$f(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\eta(x,t)}{x^2+\alpha^2}\,dx\int_0^{\infty} f(y)\,\eta(x,y)\,dy \quad (\eta(x,t) = \alpha\cos xt - x\sin xt,\ \ \alpha\geqslant 0)$$

is obtained in an analogous manner.

2. We shall now consider functions $y_k(x)$ and $z_k(x)$. Using Expressions (0.3) together with Meler and Dirichlet-Laplace formulas, we can easily show that for integral representations of Legendre polynomials [3]

$$P_k(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^{\theta}\frac{\cos(k+{}^1\!/_2)\varphi\,d\varphi}{(\cos\varphi-\cos\theta)^{1/2}} = \frac{\sqrt{2}}{\pi}\int_{\theta}^{\pi}\frac{\sin(k+{}^1\!/_2)\varphi\,d\varphi}{(\cos\theta-\cos\varphi)^{1/2}} \tag{2.1}$$

the following relations hold

$$\begin{aligned} y_k(\cos\theta) &= \frac{2\sqrt{2}}{\pi}\int_0^{\theta}\frac{\cos k\varphi\,\cos{}^1\!/_2\varphi\,d\varphi}{(\cos\varphi-\cos\theta)^{1/2}} = \frac{2\sqrt{2}}{\pi}\int_{\theta}^{\pi}\frac{\sin k\varphi\,\cos{}^1\!/_2\varphi\,d\varphi}{(\cos\theta-\cos\varphi)^{1/2}} \\ z_k(\cos\theta) &= \frac{2\sqrt{2}}{\pi}\int_0^{\theta}\frac{\sin k\varphi\,\sin{}^1\!/_2\varphi\,d\varphi}{(\cos\varphi-\cos\theta)^{1/2}} = -\frac{2\sqrt{2}}{\pi}\int_{\theta}^{\pi}\frac{\cos k\varphi\,\sin{}^1\!/_2\varphi\,d\varphi}{(\cos\theta-\cos\varphi)^{1/2}} \end{aligned} \tag{2.2}$$

From (0.3) and recurrent differential equations for Legendre polynomials we find that functions $y_k(x)$ and $z_k(x)$ satisfy

$$\begin{gathered} y_k(-x) = (-1)^{k+1} z_k(x), \quad z_k(-x) = (-1)^{k+1} y_k(x) \\ \frac{dy_k(x)}{dx} = \frac{k}{1-x}\,z_k(x), \qquad \frac{dz_k(x)}{dx} = -\frac{k}{1+x}\,y_k(x) \end{gathered} \tag{2.3}$$

The last two relations of (2.3) imply that $y_k(x)$ and $z_k(x)$ are solutions of

$$(1+x)\frac{d}{dx}\left[(1-x)\frac{dy_k}{dx}\right]+k^2y_k=0,\quad (1-x)\frac{d}{dx}\left[(1+x)\frac{dz_k}{dx}\right]+k^2z_k=0 \tag{2.4}$$

Now constructing Lõmmel integrals for these functions we obtain

$$\int\frac{y_ny_k}{1+x}dx=-\frac{1-x}{n^2-k^2}(y_n'y_k-y_k'y_n)=-\frac{ny_kz_n-ky_nz_k}{n^2-k^2}$$
$$\int\frac{z_nz_k}{1-x}dx=-\frac{1+x}{n^2-k^2}(z_n'z_k-z_k'z_n)=\frac{ny_nz_k-ky_kz_n}{n^2-k^2} \tag{2.5}$$
$$\int\frac{y_ndx}{1+x}=-\frac{z_n}{n},\qquad \int\frac{z_ndx}{1-x}=\frac{y_n}{n}$$

Let us use the following relations:

$$y_n(-1)=z_n(1)=0,\qquad y_n(1)=(-1)^{n+1}z_n(-1)=2$$

From (2.5) we obtain, that functions $y_k(x)$ and $z_k(x)$ are orthogonal on the interval $-1\leqslant x\leqslant 1$ and, that their weights are $(1+x)^{-1}$ and $(1-x)^{-1}$ respectively, i.e.

$$\int_{-1}^{1}\frac{y_ny_k}{1+x}dx=\int_{-1}^{1}\frac{z_nz_k}{1-x}dx=\begin{cases}2/n & (n=k)\\ 0 & (n\neq k)\end{cases} \tag{2.6}$$

Functions $y_n(x)$ and $z_n(x)$ are solutions of Eqs. (2.4), hence they can also be represented by hypergeometric series

$$y_n(x)=(-1)^{n+1}n(1+x)F(1+n,1-n,2,{}^1/_2(1+x))$$
$$z_n(x)=n(1-x)F(1+n,1-n,2,{}^1/_2(1-x)) \tag{2.7}$$

The hypergeometric function appearing in (2.7) was used by Tranter [4] in solving dual equations in terms of sine series.

Taking into account results obtained by Watson [5] for asymptotic expansions of the hypergeometric function when values of parameters α and β are large, we can show that as $n\to\infty$ and $|x|<1$, functions $y_n(x)$ and $z_n(x)$ tend to zero as $O(n^{-1/2})$.

From (0.3) or (2.7) we see that these functions are n-th degree polynomials (series (2.7) is truncated at the $(n-1)$th term), while (2.6) and the Weierstrass theorem imply that functions $y_k(x)$ and $z_k(x)$ form a complete and orthogonal system in the class $L_2(-1,1)$ i.e. any function $f(x)\in L_2(-1,1)$ can be represented by series

$$f(x)=\sum_{n=1}^{\infty}a_ny_n(x),\qquad f(x)=\sum_{n=1}^{\infty}b_nz_n(x) \tag{2.8}$$

whose coefficients are given, in accordance with (2.6), by

$$a_n=\frac{n}{2}\int_{-1}^{1}\frac{f(x)y_n(x)}{1+x}dx,\qquad b_n=\frac{n}{2}\int_{-1}^{1}\frac{f(x)z_n(x)}{1-x}dx \tag{2.9}$$

Examples of expansions using (2.8) which shall be utilised later, are:

$$1+\sum_{n=1}^{\infty}y_n(\cos\varphi)\cos n\beta=\begin{cases}\sqrt{2}\cos{}^1/_2\beta(\cos\beta-\cos\varphi)^{-1/2} & (\beta<\varphi)\\ 0 & (\beta>\varphi)\end{cases}$$
$$1-\sum_{n=1}^{\infty}z_n(\cos\varphi)\cos n\beta=\begin{cases}\sqrt{2}\sin{}^1/_2\beta(\cos\varphi-\cos\beta)^{-1/2} & (\beta>\varphi)\\ 0 & (\beta<\varphi)\end{cases} \tag{2.10}$$
$$\sum_{n=1}^{\infty}y_n(\cos\varphi)\sin n\beta=\begin{cases}\sqrt{2}\cos{}^1/_2\beta(\cos\varphi-\cos\beta)^{-1/2} & (\beta>\varphi)\\ 0 & (\beta<\varphi)\end{cases}$$
$$\sum_{n=1}^{\infty}z_n(\cos\varphi)\sin n\beta=\begin{cases}\sqrt{2}\sin{}^1/_2\beta(\cos\beta-\cos\varphi)^{-1/2} & (\beta<\varphi)\\ 0 & (\beta>\varphi)\end{cases}$$

Validity of these formulas when $0 < \beta$ and $\varphi < \pi$ can be confirmed using integral representations (2.2) and Formulas (2.9).

We shall now consider the dual series (0.4). Various dual series were investigated by Cooke [6], Tranter [4 and 6], Noble [7], Sneddon [8 and 9] and Srivastav [8]. Related results were also obtained in [10 and 12].

3. It is easy to see that using linear transformations and introducing new unknowns, we can write (0.4) as

$$b_0 e^{\alpha t} + \sum_{k=1}^{\infty} k b_k \chi_k(t) = f(t) \qquad (0 < t < \beta)$$

$$b b_0 e^{\alpha t} + \sum_{k=1}^{\infty} b_k \chi_k(t) = g(t) \qquad (\beta < t < \pi) \tag{3.1}$$

where b and α are given numbers, $f(t)$ is a piece-wise continuous function, and $g(t)$ is continuous and has a piece-wise continuous first derivative. The function $\chi_k(t)$ in (3.1) now has the form

$$\chi_k(t) = \begin{cases} e^{\alpha t} & (k = 0) \\ \alpha \sin kt + k \cos kt & (k = 1, 2, \ldots) \end{cases} \tag{3.2}$$

We shall introduce two operations

$$1 - \alpha \int_0^t dt, \qquad \frac{d}{dt} - \alpha$$

Applying the first one to the first Eq. and the second one to the second Eq. of (3.1), we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} (k^2 + \alpha^2) b_k \cos kt &= F_1(t) \qquad (0 < t < \beta) \\ \sum_{k=1}^{\infty} (k^2 + \alpha^2) b_k \sin kt &= G_1(t) \qquad (\beta < t < \pi) \end{aligned} \tag{3.3}$$

where

$$F_1(t) = F_1^*(t) + C, \qquad G_1(t) = \alpha g(t) - g'(t)$$

$$F_1^*(t) = f(t) - \alpha \int_0^t f(x)\, dx \qquad C = \alpha^2 \sum_{k=1}^{\infty} b_k - b_0 \tag{3.4}$$

Let us now multiply the first Eq. of (3.3) by $\cos \frac{1}{2} t (\cos t - \cos \theta)^{-1/2}$ and integrate the result in t from 0 to θ and multiply the second Eq. of (3.3) by $\cos \frac{1}{2} t (\cos \theta - \cos t)^{-1/2}$ and integrate the result in t from θ to π. The number of formal manipulations and (2.2) then yield

$$\begin{aligned} \sum_{k=1}^{\infty} (k^2 + \alpha^2) b_k y_k(\cos\theta) &= F(\theta) \qquad (0 < \theta < \beta) \\ \sum_{k=1}^{\infty} (k^2 + \alpha^2) b_k y_k(\cos\theta) &= G(\theta) \qquad (\beta < \theta < \pi) \end{aligned} \tag{3.5}$$

where

$$F(\theta) = 2C + \frac{2\sqrt{2}}{\pi} \int_0^{\theta} \frac{F_1^*(t) \cos {}^1\!/_2\, t\, dt}{(\cos t - \cos\theta)^{1/2}}, \quad G(\theta) = \frac{2\sqrt{2}}{\pi} \int_{\theta}^{\pi} \frac{G_1(t) \cos {}^1\!/_2\, t\, dt}{(\cos\theta - \cos t)^{1/2}} \tag{3.6}$$

Unknown coefficients are found from (3.5) and (2.9)

$$b_k = \frac{k}{2(k^2 + \alpha^2)} \left[\int_0^{\beta} F(\theta)\, y_k(\cos\theta) \operatorname{tg} \frac{\theta}{2}\, d\theta + \int_{\beta}^{\pi} G(\theta)\, y_k(\cos\theta) \operatorname{tg} \frac{\theta}{2}\, d\theta \right] \tag{3.7}$$

and are given in terms of the unknown magnitude C.

Simple substitution confirms the fact that (3.7) satisfies (3.3) as well as the first Eq. of (3.1), at any value of C. If the coefficient b_0 is suitably chosen, then the second Eq. of (3.1) can also be satisfied. We shall find b_0 by multiplying the second Eq. of (3.1) by $e^{\alpha t}$, integrating it with respect to t from t to π and multiplying the result by $e^{-\alpha t}$. This gives

$$bb_0 e^{\alpha\pi}\frac{\operatorname{sh}\alpha(\pi-t)}{\alpha}-\sum_{k=1}^{\infty} b_k\sin kt = e^{-\alpha t}\int_t^{\pi} g(x)\,e^{\alpha x}dx \tag{3.8}$$

Inserting b_k now from (3.7) into (3.8) and using the value of a series given by

$$\sum_{k=1}^{\infty}\frac{k y_k(\cos\theta)\sin kt}{k^2+\alpha^2}=\frac{\sqrt{2}}{\operatorname{sh}\alpha\pi}\int_0^{\theta}\frac{Q(t,\varphi)\cos{}^1\!/_2\varphi}{(\cos\varphi-\cos\theta)^{1/2}}d\varphi$$

$$Q(t,\varphi)=\begin{cases}\operatorname{sh}\alpha(\pi-t)\operatorname{ch}\alpha\varphi & (t>\varphi)\\ -\operatorname{sh}\alpha t\ \operatorname{ch}\alpha(\pi-\varphi) & (t<\varphi)\end{cases} \tag{3.9}$$

we obtain, after some transformations,

$$bb\,e^{\alpha\pi}\frac{\operatorname{sh}\alpha(\pi-t)}{\alpha}-\frac{\operatorname{sh}\alpha(\pi-t)}{\sqrt{2}\operatorname{sh}\alpha\pi}D_1+\frac{\operatorname{sh}\alpha(\pi-t)}{\sqrt{2}\operatorname{sh}\alpha\pi}\int_t^{\pi}\operatorname{ch}\alpha\varphi\cos\frac{\varphi}{2}d\varphi\times$$

$$\times\int_{\varphi}^{\pi}\frac{G(\theta)\operatorname{tg}{}^1\!/_2\theta\,d\theta}{(\cos\varphi-\cos\theta)^{1/2}}-\frac{\operatorname{sh}\alpha t}{\sqrt{2}\operatorname{sh}\alpha\pi}\int_t^{\pi}\operatorname{ch}\alpha(\pi-\varphi)\cos\frac{\varphi}{2}d\varphi\int_{\varphi}^{\pi}\frac{G(\theta)\operatorname{tg}{}^1\!/_2\theta d\theta}{(\cos\varphi-\cos\theta)^{1/2}}=$$

$$=e^{-\alpha t}\int_t^{\pi}g(x)\,e^{\alpha x}\,dx\quad(\beta<t<\pi) \tag{3.10}$$

where (3.11)

$$D_1=\int_0^{\beta}F(\theta)\operatorname{tg}\frac{\theta}{2}d\theta\int_0^{\theta}\frac{\operatorname{ch}\alpha\varphi\cos{}^1\!/_2\varphi}{(\cos\varphi-\cos\theta)^{1/2}}d\varphi+\int_{\beta}^{\pi}G(\theta)\operatorname{tg}\frac{\theta}{2}d\theta\int_0^{\theta}\frac{\operatorname{ch}\alpha\varphi\cos{}^1\!/_2\varphi}{(\cos\varphi-\cos\theta)^{1/2}}d\varphi$$

Next we shall use the transformation formula for Abelian integral Eqs.

$$\int_{\varphi}^{a}\frac{\operatorname{tg}{}^1\!/_2\theta\,d\theta}{(\cos\varphi-\cos\theta)^{1/2}}\int_{\theta}^{a}\frac{f(t)\cos{}^1\!/_2 t\,dt}{(\cos\theta-\cos t)^{1/2}}=\frac{\pi}{2\cos{}^1\!/_2\varphi}\int_{\varphi}^{a}f(t)\,dt$$

$$\int_{\beta}^{\varphi}\frac{\operatorname{tg}{}^1\!/_2\theta d\theta}{(\cos\theta-\cos\varphi)^{1/2}}\int_{\beta}^{\theta}\frac{f(t)\cos{}^1\!/_2 t\,dt}{(\cos t-\cos\theta)^{1/2}}=\frac{\pi}{2\cos{}^1\!/_2\varphi}\int_{\beta}^{\varphi}f(t)\,dt \tag{3.12}$$

$$(0\leqslant\beta<\varphi<a\leqslant\pi)$$

The validity of these formulas can be confirmed by considering the value of the following integral

$$\int_{\varphi}^{t}\frac{\operatorname{tg}{}^1\!/_2\theta d\theta}{\sqrt{(\cos\varphi-\cos\theta)(\cos\theta-\cos t)}}=\frac{\pi}{2\cos{}^1\!/_2\varphi\cos{}^1\!/_2 t}$$

Eq. (3.10) gives the following relation for b_0

$$bb_0-g(\pi)\,e^{-\alpha\pi}=\frac{\alpha e^{-\alpha\pi}}{\sqrt{2}\operatorname{sh}\alpha\pi}D_1 \tag{3.13}$$

The sum of the coefficients b_k entering (3.10) and (3.13) can be calculated using Formula

$$2\sum_{k=1}^{\infty}b_k=\int_0^{\beta}F(\theta)\,S(\theta)\operatorname{tg}\frac{\theta}{2}d\theta+\int_{\beta}^{\pi}G(\theta)\,S(\theta)\operatorname{tg}\frac{\theta}{2}d\theta=\frac{2(C+b_0)}{\alpha^2} \tag{3.14}$$

where

$$S(\theta)=\sum_{k=1}^{\infty}\frac{ky_k(\cos\theta)}{k^2+\alpha^2}=\frac{\sqrt{2}}{\operatorname{sh}\alpha\pi}\int_{\theta}^{\pi}\frac{\operatorname{sh}\alpha(\pi-\varphi)\cos{}^1/_2\varphi}{(\cos\theta-\cos\varphi)^{1/2}}\,d\varphi \tag{3.15}$$

Thus (3.13) and (3.14) define the constants b_0 and C.

In the case of dual cosine series (i.e. when $\alpha=0$ and $C=-b_0$), the above expressions simplify.

Let us now find expressions for the series entering the system (3.1). We shall use the transformation formulas (3.12) and the value of a series

$$\Sigma=\sum_{k=1}^{\infty}\frac{k\chi_k(t)\,y_k(\cos\theta)}{k^2+\alpha^2}=Q_2(\theta)+\frac{\sqrt{2}\cos{}^1/_2 t}{(\cos t-\cos\theta)^{1/2}}\qquad(t<\theta) \tag{3.16}$$

$$\Sigma=Q_2(\theta)\qquad(t>\theta)$$

$$Q_2(\theta)=-\frac{\sqrt{2}\,\alpha}{\operatorname{sh}\alpha\pi}\int_0^{\theta}\frac{Q_1(t,\varphi)\cos{}^1/_2\varphi}{(\cos\varphi-\cos\theta)^{1/2}}\,d\varphi,\quad Q_1(t,\varphi)=\begin{cases}e^{-\alpha(\pi-t)}\operatorname{ch}\alpha\varphi & (t>\varphi)\\ e^{\alpha t}\operatorname{ch}\alpha(\pi-\varphi) & (t<\varphi)\end{cases} \tag{3.17}$$

When $0\leqslant t<\beta$, we have the following expression for the second sum of (3.1)

$$\sqrt{2}\sum_{k=1}^{\infty}b_k\chi_k(t)=\cos\frac{t}{2}\left[\int_t^{\beta}\frac{F(\theta)\operatorname{tg}{}^1/_2\theta d\theta}{(\cos t-\cos\theta)^{1/2}}+\int_{\beta}^{\pi}\frac{G(\theta)\operatorname{tg}{}^1/_2\theta d\theta}{(\cos t-\cos\theta)^{1/2}}\right]-$$

$$-\sqrt{2}\,e^{\alpha t}\left[bb_0-g(\pi)e^{-\alpha\pi}\right]-\alpha e^{\alpha t}\left[\int_t^{\beta}F(\theta)\operatorname{tg}\frac{\theta}{2}\,d\theta\int_t^{\theta}\frac{e^{-\alpha\varphi}\cos{}^1/_2\varphi\,d\varphi}{(\cos\varphi-\cos\theta)^{1/2}}+\right.$$

$$\left.+\int_{\beta}^{\pi}G(\theta)\operatorname{tg}\frac{\theta}{2}\,d\theta\int_t^{\theta}\frac{e^{-\alpha\varphi}\cos{}^1/_2\varphi\,d\varphi}{(\cos\varphi-\cos\theta)^{1/2}}\right]\qquad(0\leqslant t\leqslant\beta) \tag{3.18}$$

Next we shall find an expression for a series appearing in the first Eq. of (3.1). In cases when functions $f(t)$ and $g(t)$ are given, expressions for this series can be obtained by direct substitution of (3.7) into (3.1) and use of formulas (2.10) and (3.16). In general, expression for this series is obtained in the following manner: we introduce the notation

$$h(t)=b_0e^{\alpha t}+\sum_{k=1}^{\infty}kb_k\chi_k(t)\qquad(\beta<t<\pi) \tag{3.19}$$

Then, the first Eq. of (3.1) together with (3.19) will yield, in accordance with (1.1) and (1.2),

$$b_0=\frac{2\alpha}{e^{2\alpha\pi}-1}\left[\int_0^{\beta}f(x)\,e^{\alpha x}\,dx+\int_{\beta}^{\pi}h(x)\,e^{\alpha x}\,dx\right]$$

$$b_k=\frac{2}{\pi k(k^2+\alpha^2)}\left[\int_0^{\beta}f(x)\,\chi_k(x)\,dx+\int_{\beta}^{\pi}h(x)\,\chi_k(x)\,dx\right] \tag{3.20}$$

Inserting expressions for b_k from (3.20) into the second Eq. of (3.5) we obtain, after changing the order of summation and integration,

$$\int_0^{\beta}f(x)\,S(\theta,x)\,dx+\int_{\beta}^{\pi}h(x)\,S(\theta,x)\,dx=\frac{\pi}{2}\,G(\theta)\qquad(\beta<\theta<\pi) \tag{3.21}$$

where

$$S(\theta,x)=\sum_{k=1}^{\infty}\frac{y_k(\cos\theta)\,\chi_k(x)}{k}=\sum_{k=1}^{\infty}y_k(\cos\theta)\cos kx+\alpha\sum_{k=1}^{\infty}\frac{y_k(\cos\theta)\sin kx}{k}=$$

$$=\frac{\sqrt{2}\cos{}^1/_2 x}{(\cos x-\cos\theta)^{1/2}}-1+\alpha\left(2\arcsin\frac{\sin{}^1/_2 x}{\sin{}^1/_2\theta}-x\right)\qquad(x<\theta) \tag{3.22}$$

$$S(\theta, x) = -1 + \alpha(\pi - x) \qquad (x > \theta)$$

Expression (3.22) enables us to write (3.21) as

$$\sqrt{2}\int_{\beta}^{\theta}\left(h(x) - \alpha\int_{\beta}^{x} h(y)\,dy\right)\frac{\cos{}^1/_2\,x\,dx}{(\cos x - \cos\theta)^{1/2}} = \frac{\pi}{2}G(\theta) + C_1 -$$

$$-\int_{0}^{\beta}\left(f(x) + \alpha\int_{x}^{\beta} f(y)\,dy\right)\left(\frac{\sqrt{2}\cos{}^1/_2\,x}{(\cos x - \cos\theta)^{1/2}} - 1\right)dx \qquad (\beta < \theta < \pi) \tag{3.23}$$

where

$$C_1 = \int_{\beta}^{\pi}\left(h(x) - \alpha\int_{\beta}^{x} h(y)\,dy\right)dx \tag{3.24}$$

Using (3.12) we can bring (3.23) to the form

$$\int_{\beta}^{\varphi}\left(h(x) - \alpha\int_{\beta}^{x} h(y)\,dy\right)dx = G_2(\varphi)\cos\frac{\varphi}{2} + \frac{C_1}{\pi}\,\mathrm{ar\,ctg}\left(\frac{\cos\beta - \cos\varphi}{1 + \cos\varphi}\right)^{1/2} \tag{3.25}$$

where

$$G_2(\varphi) = \frac{\sqrt{2}}{\pi}\int_{\beta}^{\varphi}\left\{\frac{\pi}{2}G(\theta) - \int_{0}^{\beta}\left(f(x) + \alpha\int_{x}^{\beta} f(y)\,dy\right)\times\right.$$

$$\left.\times\left(\frac{\sqrt{2}\cos{}^1/_2\,x}{(\cos x - \cos\theta)^{1/2}} - 1\right)dx\right\}\frac{\mathrm{tg}\,{}^1/_2\,\theta\,d\theta}{(\cos\theta - \cos\varphi)^{1/2}} \tag{3.26}$$

C_1 can be found from the second Eq. of (3.1).

Differentiating (3.24) with respect to φ we obtain the following Volterra's integral equation of second kind, from which we can find $h(x)$

$$h(x) - \alpha\int_{\beta}^{x} h(y)\,dy = G_3(x) \tag{3.27}$$

Here $G_3(x)$ is a known function

$$G_3(x) = \frac{d}{dx}\left[G_2(x)\cos\frac{x}{2}\right] + \frac{2C_1\sin{}^1/_2\,x}{\pi\sqrt{2}\,(\cos\beta - \cos x)^{1/2}} \tag{3.28}$$

From the integral Eq. (3.27) for $h(x)$, we obtain the following final expression:

$$h(x) = G_3(x) + \alpha\int_{\beta}^{x} G_3(y)\,e^{\alpha(x-y)}\,dy \tag{3.29}$$

Inserting $h(x)$ from (3.29) into (3.20) we obtain the required values of the coefficients b_k.

We see from (3.29) that function $h(x)$ has, at the point $x = \beta$, the same singularity as $G_3(x)$. This singularity can easily be obtained from (3.28) and (3.26) by integrating the latter by parts.

4. In practice one often comes across equations of the following type

$$b_0 e^{\alpha t} + \sum_{k=1}^{\infty} k b_k \chi_k(t) = f(t) \qquad (0 < t < \beta)$$

$$b b_0 e^{\alpha t} + \sum_{k=1}^{\infty} (1 - N_k) b_k \chi_k(t) = g(t) \qquad (\beta < t < \pi) \tag{4.1}$$

where α, b and N_k are given, $f(t)$ is a piece-wise continuous function and $g(t)$ is a piece-wise smooth function. We assume that N_k are bounded from above and tend to zero with increasing k, as e.g. $O(k^{-1})$.

Let us write the second Eq. of (4.1) as

$$bb_0 e^{\alpha t} + \sum_{k=1}^{\infty} c_k \chi_k(t) = g(t) + \sum_{k=1}^{\infty} N_k b_k \chi_k(t) \quad (\beta < t < \pi) \tag{4.2}$$

and apply (3.7) to the system (4.1) assuming that the right-hand side of (4.2) is known. A few simple transformations yield

$$\frac{2(k^2+\alpha^2)}{k} b_k = \sum_{p=1}^{\infty} (p^2+\alpha^2) N_p b_p I_{kp}(\beta) + \beta_k \quad (k = 1, 2, \ldots) \tag{4.3}$$

where

$$I_{kp}(\beta) = \int_\beta^\pi y_k(\cos\theta)\, y_p(\cos\theta) \operatorname{tg}\frac{\theta}{2}\, d\theta = \frac{k z_k(\cos\beta)\, y_p(\cos\beta) - p z_p(\cos\beta)\, y_k(\cos\beta)}{p^2-k^2}$$

$$n I_{nn}(\beta) = 1 + P_{n-1}P_n - \frac{1}{2}\left(P_{n-1}^2 - P_n^2\right) + 2\sin^2\beta \sum_{k=1}^{n-1} \frac{P_k(\cos\beta)\, P_k'(\cos\beta)}{k+1}$$

$$\beta_k = 2C\,\frac{z_k(\cos\beta)}{k} + \frac{2\sqrt{2}}{\pi}\left[\int_0^\beta F_2(\theta)\, y_k(\cos\theta) \operatorname{tg}\frac{\theta}{2}\, d\theta + \int_\beta^\pi G_2(\theta)\, y_k(\cos\theta)\operatorname{tg}\frac{\theta}{2}\, d\theta\right]$$

$$F_2(\theta) = \int_0^\theta \left[f(y) - \alpha\int_0^y f(x)\,dx\right] \frac{\cos y/2\,dy}{(\cos y - \cos\theta)^{1/2}}, \qquad C = \alpha^2 \sum_{k=1}^{\infty} b_k - b_0 \tag{4.4}$$

$$G_2(\theta) = \int_0^\pi [\alpha g(y) - g'(y)]\, \frac{\cos y/2\,dy}{(\cos\theta - \cos y)^{1/2}}$$

To find the coefficient b_0, we first insert the values of b_k obtained from (4.3) into (4.2). Then we follow the procedure similar to that used in the derivation of (3.13), also using the value of (3.16) and the integral

$$\int_\varphi^\pi \frac{y_k(\cos\theta)\operatorname{tg}{}^1\!/_2\theta\, d\theta}{(\cos\varphi - \cos\theta)^{1/2}} = \frac{\sqrt{2}}{\cos{}^1\!/_2\varphi}\, \frac{\cos k\varphi - (-1)^k}{k} \tag{4.5}$$

which is obtained from the first formula of (2.2) by considering it as an integral equation of the type (3.12). Eqs. (4.2) and (4.3) yield

$$bb_0 e^{\alpha\pi} + \frac{\alpha}{\sqrt{2}\operatorname{sh}\alpha\pi} \sum_{p=1}^{\infty} (p^2+\alpha^2)\, b_p N_p \int_0^\beta y_p(\cos\theta)\operatorname{tg}\frac{\theta}{2}\, d\theta \int_0^\theta \frac{\operatorname{ch}\alpha\varphi \cos{}^1\!/_2\varphi}{(\cos\varphi - \cos\theta)^{1/2}}\, d\varphi +$$

$$+ \frac{\sqrt{2}C}{\operatorname{sh}\alpha\pi} \int_0^\beta \frac{\operatorname{sh}\alpha\varphi \sin{}^1\!/_2\varphi}{(\cos\varphi - \cos\beta)^{1/2}}\, d\varphi - \frac{2\alpha D_2}{\pi\operatorname{sh}\alpha\pi} - g(\pi) = 0 \tag{4.6}$$

where D_2 is given by a formula similar to (3.11), in which functions $F(\theta)$ and $G(\theta)$ are replaced with $F_2(\theta)$ and $G_2(\theta)$ from (4.4).

Let us introduce into (4.3) and (4.6) new unknowns together with the following notation

$$X_0 = bb_0 e^{\alpha\pi}, \qquad X_k = \frac{2(k^2+\alpha^2)}{k} b_k, \qquad a_{kp} = \frac{pN_p}{2} I_{kp}(\beta)$$

$$a_{0p} = -\frac{\alpha p N_p}{2\sqrt{2}\operatorname{sh}\alpha\pi} \int_0^\beta y_p(\cos\theta)\operatorname{tg}\frac{\theta}{2}\, d\theta \int_0^\theta \frac{\operatorname{ch}\alpha\varphi\cos{}^1\!/_2\varphi}{(\cos\varphi-\cos\theta)^{1/2}}\, d\varphi \tag{4.7}$$

$$\beta_0 = g(\pi) + \frac{2\alpha D_2}{\pi\operatorname{sh}\alpha\pi} - \frac{\sqrt{2}C}{\operatorname{sh}\alpha\pi}\int_0^\beta \frac{\operatorname{sh}\alpha\varphi\sin{}^1\!/_2\varphi}{(\cos\varphi-\cos\beta)^{1/2}}\, d\varphi$$

Then (4.3) will become

$$X_k = \sum_{p=1}^{\infty} a_{kp} X_p + \beta_k \qquad (k = 0,\ 1,\ 2,\ \ldots) \tag{4.8}$$

Now we shall utilise the fact that N_k entering (4.1) and (4.8) are bounded from above and tend to zero with $k \to \infty$ as $O(k^{-1})$ to prove that the infinite system (4.8) is quasi-completely regular.

Taking into account the inequalities

$$|N_k| \leqslant \frac{m}{k}, \quad |y_k(x)| < \frac{2}{\sqrt{k}}, \quad |z_k(x)| < \frac{2}{\sqrt{k}} \quad \text{when } |x| < 1 - \varepsilon \tag{4.9}$$

we obtain the following estimate for the sum of moduli of the coefficients accompanying the unknowns

$$\sum_{p=1}^{\infty} |a_{kp}| = \frac{1}{2} \sum_{p=1}^{\infty} p\, |N_p I_{kp}(\beta)| < \frac{m}{k} + \frac{2m}{\sqrt{k}} \sum_{\substack{p=k \\ p\neq k}}^{\infty} \frac{1}{\sqrt{p}\,|p-k|} =$$

$$= \frac{m}{k} + \frac{2m}{\sqrt{k}} \left[\frac{1}{2} \sum_{p=1}^{k-1} \frac{\sqrt{k-p} + \sqrt{p}}{p(k-p)} + \sum_{p=1}^{\infty} \frac{1}{p\sqrt{p+k}} \right] \leqslant \frac{m}{k} +$$

$$+ \frac{2m}{\sqrt{k}} \left[\frac{\sqrt{k-1}+1}{k-1} + \frac{1}{\sqrt{k}} \ln \frac{(\sqrt{k} + \sqrt{k-1})^4 (\sqrt{k}-1)}{\sqrt{k}+1} + \frac{1}{\sqrt{k+1}} \right] \leqslant \frac{5 + 4\ln 4k}{k} m$$

which tends to zero with increasing k. This means that, beginning from some number k_0, we shall have

$$\sum_{p=1}^{\infty} |a_{kp}| < 1 - \varepsilon \qquad (k \geqslant k_0) \tag{4.10}$$

i.e. the infinite system (4.8) is quasi-completely regular. Value of k_0 depends on the values of N_k and can easily be found in each particular case.

Using the previous assumptions concerning $f(t)$ and $g(t)$ we can show (taking (4.9) into account) that independent terms of (4.8) are bounded from above and tend to zero with increasing k, as $\beta_k = 0\ (k^{-3/2})$.

Unknown coefficients X_k (or b_k) entering the last relation of (4.4), the latter assuming by virtue of (4.7) the form

$$C = \frac{\alpha^2}{2} \sum_{k=1}^{\infty} \frac{k X_k}{k^2 + \alpha^2} - \frac{X_0 e^{-\alpha\pi}}{b} \tag{4.11}$$

can be found from the quasi-completely regular infinite system of linear equations (4.8) and given in terms of a constant C, since the free terms β_k of this system depend on C. Inserting the values of X_k obtained from (4.8) into (4.11) and solving the obtained relation for C, we obtain its value.

Having found X_k we can determine the series entering (4.1). Since X_k tend to zero when $k \to \infty$ as $X_k = 0\ (k^{-3/2})$, the sum of the second series of (4.1) will be a bounded and continuous function (the series converges absolutely) which can be computed by numerical methods. The first series of (4.1) does not converge absolutely and its sum is, in general, a discontinuous function which becomes infinite at the point $t = \beta + 0$.

To separate the singularity (its principal part) of this series, we shall insert into it the values of X_k obtained from (4.8)

$$\sum_{k=1}^{\infty} k b_k \chi_k(t) = \frac{1}{4} \sum_{p=1}^{\infty} p N_p X_p \sum_{k=1}^{\infty} \frac{k^2 I_{kp}(\beta) \chi_k(t)}{k^2 + \alpha^2} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{k^2 \beta_k \chi_k(t)}{k^2 + \alpha^2} \tag{4.12}$$

Let us use the representations

$$I_{kp}(\beta) = -\frac{z_k(\cos\beta)\, y_p(\cos\beta)}{k} + \frac{p}{k}\int_\beta^\pi z_k(\cos\theta)\, z_p(\cos\theta)\operatorname{ctg}\frac{\theta}{2}\, d\theta$$

$$\beta_k = \frac{2\sqrt{2}}{\pi k}\left[F_2(\beta) - G_2(\beta) + \frac{\pi}{\sqrt{2}}C\right] z_k(\cos\theta) - \frac{2\sqrt{2}}{\pi k}\left[\int_0^\beta F_2'(\theta)\, z_k(\cos\theta)\, d\theta + \right.$$

$$\left. + \int_\beta^\pi G_2'(\theta)\, z_k(\cos\theta)\, d\theta\right] \quad (k = 1, 2, \ldots) \tag{4.13}$$

and the following value of the series:

$$\Sigma_2 = Q_3(\theta) \qquad (t < \theta)$$

$$\Sigma_2 = \sum_{k=1}^{\infty} \frac{k z_k(\cos\theta)\, \chi_k(t)}{k^2 + \alpha^2} = -\frac{\sqrt{2}\sin{}^1\!/_2 t}{(\cos\theta - \cos t)^{1/2}} + Q_3(\theta) \qquad (t > \theta)$$

$$Q_3 = \frac{\sqrt{2}\alpha}{\operatorname{sh}\alpha\pi}\int_\theta^\pi \frac{Q_1(t,\varphi)\sin{}^1\!/_2 t}{(\cos\theta - \cos\varphi)^{1/2}}\, d\varphi \tag{4.14}$$

where $Q_1(t, \varphi)$ is given by (3.17); from (4.12) we obtain for $\beta < t < \pi$

$$\sum_{k=1}^{\infty} k b_k \chi_k(t) = \frac{M\sin{}^1\!/_2 t}{(\cos\beta - \cos t)^{1/2}} + \varphi(t) \qquad (\beta < t < \pi) \tag{4.15}$$

where $\varphi(t)$ is a bounded and continuous function easy to determine in each particular case, and M is

$$M = \frac{1}{2\sqrt{2}}\sum_{k=1}^{\infty} k N_k X_k y_k(\cos\beta) - \sqrt{2}\,C - \frac{2}{\pi}F_2(\beta) + \frac{2}{\pi}G_2(\beta) \tag{4.16}$$

In conclusion we shall note that dual series-equations in $\eta_k(t)$, $y_k(x)$ and $z_k(x)$ as well as dual integral equations in $\chi(x, t)$ and $\eta(x, t)$ can be solved in an analogous manner.

For example, to solve dual equations in $y_k(x)$ we have

$$\sum_{k=1}^{\infty} k a_k y_k(\cos\theta) = f(\theta) \quad (0 < \theta < \beta), \qquad \sum_{k=1}^{\infty} a_k y_k(\cos\theta) = g(\theta) \quad (\beta < \theta < \pi) \tag{4.17}$$

where functions $f(\theta)$ and $g(\theta)$ satisfy the same requirements as those in (4.1). Let us multiply the first Eq. of (4.17) by $\operatorname{tg} {}^1\!/_2\theta(\cos\theta - \cos\varphi)^{-1/2}$ and integrate it in θ from 0 to φ, and the second Eq. of (4.17) by $\operatorname{tg} {}^1\!/_2\theta(\cos\varphi - \cos\theta)^{-1/2}$ integrating it then in θ from φ to π. Utilising the values of integrals

$$\int_0^\varphi \frac{y_k(\cos\theta)\operatorname{tg}{}^1\!/_2\theta\, d\theta}{(\cos\theta - \cos\varphi)^{1/2}} = \sqrt{2}\,\frac{\sin k\varphi}{k\cos{}^1\!/_2\varphi}, \quad \int_\varphi^\pi \frac{y_k(\cos\theta)\operatorname{tg}{}^1\!/_2\theta\, d\theta}{(\cos\varphi - \cos\theta)^{1/2}} = \sqrt{2}\,\frac{\cos k\varphi - (-1)^k}{k\cos{}^1\!/_2\varphi} \tag{4.18}$$

obtained from (2.2) by considering them as integral equations of the type (3.12) we obtain, from (4.17),

$$\sum_{k=1}^{\infty} a_k \sin k\varphi = f_1(\varphi) \quad (0 < \varphi < \beta), \qquad \sum_{k=1}^{\infty} a_k \sin k\varphi = g_1(\varphi) \quad (\beta < \varphi < \pi) \tag{4.19}$$

$$\begin{aligned} f_1(\varphi) &= \frac{1}{\sqrt{2}}\cos\frac{\varphi}{2}\int_0^\varphi \frac{f(\theta)\operatorname{tg}{}^1\!/_2\theta\, d\theta}{(\cos\theta - \cos\varphi)^{1/2}} \\ g_1(\varphi) &= -\frac{1}{\sqrt{2}}\frac{d}{d\varphi}\left[\cos\frac{\varphi}{2}\int_\varphi^\pi \frac{g(\theta)\operatorname{tg}{}^1\!/_2\theta\, d\theta}{(\cos\varphi - \cos\theta)^{1/2}}\right] \end{aligned} \tag{4.20}$$

Relations (4.19) yield the following values of a_k

$$\frac{\pi}{2} a_k = \int_0^\beta f_1(\varphi) \sin k\varphi \, d\varphi + \int_\beta^\pi g_1(\varphi) \sin k\varphi \, d\varphi \qquad (4.21)$$

and

$$\frac{\pi}{2\sqrt{2}} \sum_{k=1}^{\infty} a_k y_k(\cos\theta) = \int_\theta^\beta \frac{f_1(\varphi)\cos{}^1/_2\varphi \, d\varphi}{(\cos\theta - \cos\varphi)^{1/2}} + \int_\beta^\pi \frac{g_1(\varphi)\cos{}^1/_2\varphi \, d\varphi}{(\cos\theta - \cos\varphi)^{1/2}} \qquad (0 \leqslant \theta \leqslant \beta)$$

$$\frac{\pi}{2\sqrt{2}} \sum_{k=1}^{\infty} k a_k y_k(\cos\theta) = \operatorname{ctg}\frac{\theta}{2}\frac{d}{d\theta}\left[\int_0^\beta \frac{f_1(\varphi)\sin{}^1/_2\varphi \, d\varphi}{(\cos\varphi - \cos\theta)^{1/2}} + \int_\beta^\theta \frac{g_1(\varphi)\sin{}^1/_2\varphi \, d\varphi}{(\cos\varphi - \cos\theta)^{1/2}}\right]$$

$$(\beta < \theta \leqslant \pi)$$

Here we have used the formulas (2.8) to (2.10) and (4.21).

BIBLIOGRAPHY

1. Babloian, A.A., Solution of the plane problem of the theory of elasticity for an annular sector, in terms of stresses. Izv. Akad. Nauk ArmSSR, Ser. Fiz-matem. nauk, Vol. 15, No. 1, 1962.
2. Abramian, B.L. and Babloian, A.A., On a contact problem connected with the torsion of a hollow semisphere. PMM Vol. 26, No. 3, 1962.
3. Lebedev, N.N., Special Functions and Their Applications. 2nd edition, M.-L., Fizmatgiz, 1963.
4. Tranter, C.J., Dual trigonometrical series. Proc. Glasgow Math. Assoc., Vol. 4, No. 2, 1959.
5. Watson, G.N., Theory of Bessel Functions. Cambridge Univ. Press, 1944.
6. Cooke, J.C., and Tranter, C.J., Dual Fourier – Bessel series. Quart. J. Mech. and Appl. Math., Vol. 12, No. 3, 1959.
7. Noble, B., Some dual series equations involving Jacobi polynomials. Proc. Cambridge Philos. Soc., Vol. 59, No. 2, 1963.
8. Sneddon, I.N. and Srivastav, R.P., Dual series relating; Dual relations involving Fourier – Bessel series. Proc. Roy. Soc. Edinburgh, ser. A, Vol. 66, p. 3., 1962-1963.
9. Sneddon, I.N., Dual Equations in Elasticity. Coll. Applications of the theory of functions to the mechanics of continuous media. (Russian translation) Trudy Mezhdun. simpoz. Tbilisi, 1963, Vol. 1, M., published by "Nauka", 1965.
10. Minkov, I.M., On some functional equations. PMM Vol. 24, No. 5, 1960.
11. Tseitlin, A.I., On the methods of dual integral equations and dual series and their application to problems of mechanics. PMM Vol. 30, No. 2, 1966.
12. Babloian, A.A., Solution of some dual series. Dokl. Akad. Nauk ArmSSR, Vol. 39, No. 3, 1964.

Translated by L.K.